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as Professor of Mathematics in the Marine corps. He was named a Counsellor of State in 1800, and died in 1826.

Tetens himself, although from writing in the German language he claims to belong to German literature, may still be more correctly considered as a *Dano-German*, having been born in Schleswig, and having resided either in that duchy or in Copenhagen, during the whole of his life. As six years elapsed between the publication of his work, and the date of Dr. Price's death, it is very likely that Tetens (regard being had to his express desire to instruct the British public in some measure) may, through Mr. Oeder or otherwise, have taken means to communicate his researches to Dr. Price. From Morgan's Biography of the latter it will be perceived that he had not only a superfluity of occupation in those years, but that he laboured under increasing ill health, which interrupted the publication of the fourth edition of his work, a matter he had much at heart, and for which he was constantly collecting materials. The consequence of his illness was that the second volume of the work (which would have contained any *additional particulars* on foreign subjects) was entrusted to the care of Mr. William Morgan. (See the latter's "Memoirs of Dr. Price," pp. 173 and 174).

Are Dr. Price's MS. collections yet extant? if so, they would perhaps clear up this point.

On a general method of approximation to the Values of Annuities and Assurances, for long terms of years, depending on One or Two Lives.

By Samuel Brown, Esq., Actuary of the Mutual Life Assurance Society.

(*Read before the Institute of Actuaries, 24th June,* 1850.)

The discovery of some general Law of Mortality, by means of which the values of the contingencies, depending upon human life, might be expressed in some simple formulæ, without the necessity of performing the numerous calculations which are required even for the chances of a single life, and still more in the various cases, in which two or more lives are involved, has occupied the attention and labours of many scientific men. But the result has not hitherto been satisfactory, in spite of the skill with which their investigations have been conducted, and the unwearied patience which they have brought to the task. The failure has probably arisen from the want of a sufficient number of observations, the authenticity of which could be relied upon; or for want of that accuracy in the data, without which it is in vain to attempt to find the traces of a regular law, disturbed as it must be by the peculiar circumstances under which the observations were made, or by the ignorance, and, in some cases it may be, by the wilful errors of those who have been employed to collect the facts. To take, for instance, the census of the whole kingdom, which, if conducted on a uniform plan, at regular intervals, and with the births and deaths from year to year carefully observed according to locality, ought to afford to the scientific enquirer the means of arriving at the general Law of Mortality, if any such existed—can we turn with any confidence to the enumeration of the Living, or the records of Births and Deaths previous to the Registration Act, under which, in 1838, the present improved system of observation was commenced? Before that period, the censuses were taken, either without the ages of the living, or with them at intervals so long, that intervening circumstances might occur to render the whole comparison incorrect. The deficiency in the number of burials as entered in the Parish Registers was so great, and consequently the correction to be made by the computer allowed of a diversity of opinion so wide as to detract very much from the value of the conclusions drawn from the facts. The rough estimate of ages, which even under existing improvements is all that can be expected, when so many millions of persons are to be enumerated in a single day, and the effect of which is very visible in the much larger proportion, represented as living at particular ages, than are at all in agreement with the previously recorded mortality, increases the difficulty of obtaining any true results from such uncertain data.

The registration of the births, which is an important element in ascertaining whether a population is increasing or decreasing, is very deficient even still, and has been noticed by the Registrar General with a view to its correction. Mr. Edmonds, to whose recent valuable paper I shall again refer, considers "that in order to obtain the true number of births, there is required an increase of two-ninth parts of the baptisms and an increase of one-ninth part of the births registered under the new act." With all these defects and inconsistencies it is not surprising that very little assistance has been afforded in ascertaining the true Law of Mortality prevailing in Great Britain from the censuses which have been hitherto taken. Some light will, no doubt, be thrown on the subject, when the Decennial Census to be taken next year shall have been completed with the ability which the Registrar General and his assistants have exhibited in the last ten Reports, and when the results, obtained on a uniform plan, shall be compared with those deduced from the census of 1841.

Perhaps it may be suggested, that if the deficiences and errors to which a public census must always be liable, cannot be easily overcome, we might have recourse with more confidence to the recorded Experience of Life Assurance Companies, where the facts at least may be most accurately observed, where the ages are proved with legal precision, the enumeration of the living at every age made from the books without the possibility of misrepresentation; and the causes of the deaths certified in the presence of witnesses, with an accuracy and minuteness which leaves nothing to be desired in the collection of the observations. There can be no doubt that we have here all the information that can be needed; but there are other circumstances which, without diminishing the accuracy of the data thus registered, detract at present from their value as matters of science. The first of these qualifying circumstances is, that from the method of inquiry, the whole of the lives under observation ought to be carried forward, from year to year, till the last survivor of a particular class expires. Otherwise, we shall be falling into the unphilosophical error of assuming that the future experience is to resemble the past, when the very point at issue is to know in what degree the future will be modified by the variations from existing laws, which we may find to prevail in the early history of a company. The length of time required for this correct series of observations is so great, (extending at least from the average age at which parties enter the society to the extreme limits of the duration of human life,) that very few societies are yet in a condition to furnish the requisite data, even if they had always preserved the materials in a proper form for inquiry, and were willing to incur the expense and labour of making them public. Nor would the mere enumeration of all the mixed classes of lives, as existing at the respective ages, be sufficient for scientific purposes, though it may answer well enough for ascertaining the relative mortality in the society, as compared with that of the table from which the premiums are deduced.

The influence of the Selection of Lives for assurance has been supposed by some writers to terminate within a very short period from the admission into the society; but I am inclined to think, that wherever the facts are sufficiently numerous to give average results, the influence of selection will be felt by a diminution of the mortality below the average before, and increase of it above the average after a period (varying according to the age at entry, or the skill and care shown in the examination), throughout the whole period of life under observation. There are no printed tables, except those of the Equitable Society, in which the numbers of the assured are subdivided into classes from periods of admission, with sufficient minuteness to warrant the deduction of any general law of the effect of Selection at present; but a table, which I have given elsewhere, reduced to the mortality per cent. from the one, which will be found in Mr. Arthur Morgan's preface to the Tables printed in 1834, shows strong indications that additional observations would confirm the interesting fact noticed above.* No one can deny the importance of such a conclusion, (if it be a correct one,) to new companies, amongst the members of which a general impression prevails, that the favourable results of early years only indicate a great improvement in the public health, and consequently undiminished profits, whereas, the conclusion which I should draw from the same facts is, that the mortality in advancing years will be proportionably increased above the average of the older ages. If the valuable materials collected by the Committee of Actuaries, and published as the results of the experience of companies, could be enlarged by the addition of the increasing body of facts observed since the former publication, and the classes of assured separated as above proposed, perhaps from the large number of observations, some very important general laws might be discovered; though I should still be inclined to believe, that on the point now referred to, no decisive opinion could be formed for another thirty or forty years to come. Time, as well as numbers, is a necessary element in an investigation in which the future history of a society is concerned. Neither must it be forgotten, that the regard lately paid to sanitary measures, may, amongst the classes who have suffered the most from the want of government attention to their health and physical condition, act contrary to the interests of societies, which are now extending their benefits to the labouring population. It is possible, (and in my own opinion, not improbable,) that the very creditable sanitary precautions now urged on the public may be the means of preserving numbers of children, through a sickly period of childhood, to live to manhood, it is true, but to enjoy perhaps only a weakly and precarious state of health at a time of life when they begin more to feel and to require the advantages of benefit clubs, and similar institutions. This subject, however, would lead us far away from the object of the present article, and may be resumed on some future occasion.

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* The recent paper by Mr. Higham, " On the Value of Selection amongst Assured Lives," read before the Institute of Actuaries, March 25, 1850, and ordered by the Council to be printed, contains some newly-formed Tables of Expectation, amongst the classes of Lives, selected at different ages, which are most skilfully prepared, and merit particular attention.

I have ventured to draw the notice of the members of the Institute to these reasonings, (though many must already have found their attention occupied by them,) because they serve to explain the difficulties which have hitherto prevented the discovery of a general law of mortality prevailing throughout the whole term of years to which the life of man extends. They at the same time encourage the exertions of the statist and philosopher to persevere in investigations which are of the deepest interest to science and humanity, by the reflection, that every day is not only adding to our recorded experience, but gives it a value which it did not before possess, by the increased duration of the period over which the enquiries may be pursued.

In the meantime the question arises, whether we cannot by some simple formulæ, applied to the original facts, approximate to the results of the complicated and numerous calculations which are necessary for ascertaining the probabilities of living and dying in cases where one or more lives are concerned, or in the still more laborious processes, involving the improvement of money during a life or lives at different rates of interest. In the latter case it will appear almost on inspection, that any hypothesis assuming the law of a geometrical progression, increasing in the ratio of mortality from age to age, or decreasing in the numbers surviving from age to age, will have a great advantage over any other method that could be proposed, if, in other respects, it could be presented in a form sufficiently simple to be readily understood, and easily reduced to practice, and if it could be made to represent the original facts with sufficient accuracy. Scarcely any authentic tables of observation however have been found, which allow of this hypothesis being used at only one rate of progression throughout the whole series of ages in the table. At some portion of the period between fifty and sixty years of age, a very rapid increase in the rate of mortality is found to take place, which continues with more or less regularity, but always very much above the earlier portion of the table, to the last age of the living. The decrease of the numbers living at each succeeding age, out of a given number living at any one age, is only found to be in geometrical progression for a short period of years, and this necessitates a similar calculation to be frequently repeated, and adds much to the labour of obtaining the value of an annuity upon a life for a long term of years, or for the term by which the duration of life according to the observations is limited. In the first of the two valuable and learned papers which Mr. Gompertz gave to the world through the Transactions of the Royal Society in 1820, he introduces this hypothesis, for small intervals, by showing that "we may take an interval m from n so small, whatever be the real constitution of the function, that the number of the living during that interval shall decrease so as to form a geometrical progression, very nearly, whilst the portions of time increase in an arithmetical progression, and that the decrements of living are also in consequence, very nearly in geometrical progression; and we may moreover, in this, as in the former case, (viz., that the intervals of age may be taken so small, that whatever the law of mortality may be, during any portion of any the same interval, the decrements may be considered proportional to the time,) accommodate the function so as to be accurate at the two extremes."

The second paper is a continuation of the same subject, published in the Philosophical Transactions in 1825, in which, from the examination of different tables, he is led to believe "that the law of geometrical progression pervades in an approximate degree large portions" of such tables. In a subsequent part of the same paper, Mr. Gompertz has enunciated what may, with greater probability, be deemed to represent the general law of mortality, though the circumstances by which it may be modified are so numerous, that it will require great care and attention on the part of those who would apply the formulæ deduced therefrom, to fix the constants, so as to represent accurately the rate of mortality amongst the class of persons for whom the calculations may be required. Mr. Gompertz remarks, "it is possible, that death may be the consequence of two generally coexisting causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or an increased inability to withstand destruction. If, for instance, there be a number of diseases to which the young and old are equally liable, and likewise which should be equally destructive, whether the patient be young or old, it is evident, that the deaths amongst the young and old by such diseases, would be exactly in proportion of the number of young to the old, provided those numbers were sufficiently great for chance to have its play; and the intensity of mortality might then be said to be constant, and were there no other diseases but such as those, life of all ages would be of equal value, and the number of living and dying, from a certain number living at a given earlier age, would decrease in geometrical progression as the age increased by equal intervals of time; but if mankind be continually gaining seeds of indisposition, or in other words, an increased liability to death, which appears not to be an unlikely supposition with respect to a great part of life, (though the contrary appears to take place at certain periods,) it would follow, that the number of living out of a given number of persons at a given age, at equal successive increments of age, would decrease in a greater ratio than the geometrical progression; and then the chances against the knowledge of any one having arrived to certain defined terms of old age, might increase in a much faster progression, notwithstanding there might still be no limit to the age of man."

"If the average exhaustions of a man's power to avoid death were such, that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of those intervals, then at the age x, his power to avoid death or the intensity of his mortality, might be denoted by aq^x, a and q being constant quantities; and if L_x be the number living at the age x, we shall have $aL_x \times q^x \dot{x}$ for the fluxion of the number of deaths

$$= -(L_x)^{\cdot} \therefore abq^x = -\frac{\dot{L}_x}{L_x} \therefore abq^x = -\text{hyp. log. of } b \times \text{hyp. log. of } L_x$$

and putting the common log. of $\frac{1}{b}$ × square of the hyp. log of 10 $= c$, we have $c.q^x =$ com. log. of $\frac{L_x}{d}$; d being a constant quantity, and therefore L_x or the number of persons living at the age $x = d.\overline{g}|^{q^x}$ g being put for the number whose common log. is c."

The following sentence deserves attention, as summing up the reasons for accepting an hypothesis of this nature, founded as it is on higher grounds than most of those which have been previously advocated:—" This equation between the number of living and the age becomes deserving of attention, not in consequence of its hypothetical deduction, which in fact is congruous with many natural effects, as for instance the exhaustions of the receiver of an air-pump by strokes repeated at equal intervals of time, but is deserving of attention because it appears corroborated during a long portion of life by experience, as I derive the same equation from various published Tables of Mortality during a long period of man's life, which experience therefore proves, that the hypothesis approximates to the law of mortality during the same portion of life; and in fact *the hypothesis itself was derived from an analysis of the experience here alluded to.*"

Mr. Gompertz then proceeds to explain the various tables to which he has applied this hypothesis; and as the volume in which the above article appears may not be in the hands of every younger student, for whom the paper which I have now the honour of laying before the members of the Institute is more especially designed, I subjoin the formula which he has deduced to obtain the number living at any given age x by different tables.

$(\lambda\, L_x)$

Ages 15 to 55	Northampton Table	Log. of living at age x	$= 3\cdot9264$ —	Number whose log. is	$(\overline{1}\cdot11556+\cdot011213x)$
15 to 55	Deparcieux Table in Baily's Ann.	„	$= 3\cdot1557$ —	„	$(\overline{1}\cdot25095+\cdot00704\ x)$
10 to 50	Swedish Mortality in Baily's Ann.	„	$= 3\cdot8703$ —	„	$(\overline{2}\cdot82861+\cdot013145x)$
10 to 60	Milne's Carlisle Table	„	$= 3\cdot88631$ —	„	$(\overline{2}\cdot88126+\cdot0126\ x)$
60 to 100	Ditto	„	$= 3\cdot79657$ —	„	$(\overline{3}\cdot74767+\cdot02706\ x)$
20 to 60	Equitable Experience	„	$= 3\cdot813$ —	„	$(\overline{2}\cdot6051+\cdot0149875x)$

The last is deduced from the rough comparison with the Northampton Table, which Mr. Morgan appended to a note to the sixth Edition of Dr. Price's Works, and the results by Mr. Gompertz' hypothesis somewhat differ from the Tables of Equitable Experience as printed by the Society.

In the second portion of this interesting paper, Mr. Gompertz proceeds to give examples of the use of the Tables of what are called "accommodated ratios," which are intended to expedite the operations required, from assuming the number of persons living at equal intervals of successive ages to be in geometrical progression, and the periods taken sufficiently short to permit this assumption to be a near approximation to the truth. It is unnecessary to enter into any description of these Tables,* which are extremely useful where an approximation to the values of Annuities on several lives may be required for short periods, my object being rather to explain a method by which, assuming the same law of geometrical progression in the ratio of mortality at successive equal intervals of age, we may, without any tables at all, obtain the values of Annuities and Assurances, for one or two lives, for a term of thirty or forty years, provided the last year of this term falls within what Mr. Edmonds designates the period of manhood. This period very nearly corresponds with those for which Mr. Gompertz has given the formulæ, which I have before quoted, and according to different Tables will be found to vary from 10 or 12 to about 55 or 60 years of age.

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* It may be observed that the use of these Tables will be found to shorten very much some portion of the work in the approximation now proposed. It will be seen on inspection by those who know the formation of them.

It is known, I believe, to most of those who take an interest in the subject of Life Assurance, that Mr. Edmonds considers he has discovered the true ratio of mortality at each of the three periods, marked out by nature, and easily discoverable in all tables of observations, as those of infancy, manhood, and old age, the first being marked by a decreasing ratio of mortality of about 32½ per cent. per annum to about 8 years of age; the second by an increasing ratio of 3 per cent. from 12 to 55 years of age; and the last, by the more rapidly increasing ratio of 8 per cent. in each successive age of life to the close of the tables. Mr. Edmonds, in his work published in 1832, contends, "that the new theory is universally true. All valuable observations made in Europe concur in proving its truth, and recent extensive and accurate observations made on the Jamaica slave population, of African parentage, are in conformity with it. Whence the conclusion is warrantable, that the new theory is equally applicable to the lowest as well as to the highest grade of humanity, and to the inhabitants of tropical as well as of polar regions." Since the publication of that work, Mr. Edmonds has written a number of valuable papers to prove the truth of the numerical law of his theory, examining and comparing all the authentic observations, the accuracy of which could be sufficiently relied upon. In the *Lancet* of 9th and 16th March of this year, will be found a very important article on this subject, comparing the results deduced from the Registrar General's Reports of deaths, during the seven years 1838 to 1844, and the enumeration of the living in 1831, not only with his Theoretical Tables, but with the previous imperfect observations of the living and the deaths in Great Britain, to which I have before alluded. There are so few collections of facts which have been made with the care and accuracy, and with the minute statement of particulars to give them the desirable authority for deductions of so much importance, and the corrections required must be so open to the arbitrary adjustment of individual skill, that the discovery of a law, so extensive as that asserted to govern the mortality of the whole human race, may require with many minds much more proof than can at present be offered. The remarkable paper however by Mr. Gompertz, "On the nature of the function expressing the law of human mortality," and the practical researches of Mr. Edmonds, which have in almost every case ended in proving the truth of the theory by actual numerical results, deserve more investigation than they have hitherto received. Perhaps the higher class of mathematicians, to whom Mr. Gompertz' paper has been principally confined, may have shrunk from the laborious compilation of facts, or the troublesome calculations of those already recorded, whilst the data were so uncertain or so ill arranged to facilitate enquiry. If this attempt to show the usefulness of an approximation to the true results of the theory, obtained by the very elementary rules of algebra, may draw attention to the theory itself, and induce some of the members of the Institute, who may have leisure at command, to examine, on the plan pursued by Mr. Edmonds, the collections of facts already made, with a view to test its truth or error, it may tend either still further to establish its correctness or to lead to the true law which it is so desirable to ascertain.

Before concluding this introduction, it may be remarked, that Mr. Galloway, in the formation of the tables from the Amicable Experience, has used Mr. Gompertz' formula as a final correction in the 9th column of Table III, employing the same constants from age 45 to 93, and the formula which gives the number of survivors at any intermediate age, (y_{45} being the number living at age 45) is

$$\log.\ y_{45+x} = 4.08966 - \left\{ \begin{matrix} \text{the number} \\ \text{whose log. is} \end{matrix} \right\} 8.95259 + .0296275\ x$$

In Mr. Farren's recently published and very useful "Life Contingency Tables," Part I., Mr. Gompertz' formula is also applied to the continuation of the corrected table, at ages where the admissions are too few, to allow of the supposition, that the same mixed generic law will continue to prevail in the general mortality," as prevailed previously; and in Section I. will be found a full account of Mr. Gompertz' theory, and the method of obtaining his formulæ.

Method of approximating, from original Tables of Observations, to the Values of Life Contingencies (whether involving or not the Interest of Money,) on any Single or on any Two Joint Lives for long Terms of Years.

I. Probability of a single life at any given age living any number of years n.

Let m = the ratio of mortality to an unit of life, at the lowest age in the Table to which the constant will apply, which I will call the standard age.

x = the common ratio, with which the mortality increases from year to year.

p = the number of years difference between the standard age and the given age.

Then will mx^p represent the rate of mortality at any age, p years older than the standard age.

The following Table will exhibit the respective probabilities of dying in a year and living to the end of a year, at the standard and the following ages:

Age.	Probability of dying in a year.	Probability of living a year.
When $p = 0$	$mx^p = m$	$1 - m$
$p = 1$	$mx^p = mx$	$1 - mx$
$p = 2$	$mx^p = mx^2$	$1 - mx^2$
$p = 3$	$mx^p = mx^3$	$1 - mx^3$
$p = n$	$mx^p = mx^n$	$1 - mx^n$

The probabilities of surviving to the end of 1, 2, 3, n years, from the given age p, will be respectively,

Probability of a life aged p years older than the standard age surviving to age

$$\begin{aligned}
p+1 &= \left(1-mx^{p}\right)\\
p+2 &= \left(1-mx^{p}\right)\cdot\left(1-mx^{p+1}\right)\\
p+3 &= \left(1-mx^{p}\right)\cdot\left(1-mx^{p+1}\right)\cdot\left(1-mx^{p+2}\right)\\
p+n &= \left(1-mx^{p}\right)\cdot\left(1-mx^{p+1}\right)\cdot\text{\&c.}\,\ldots\ldots\left(1-mx^{p+n-1}\right)
\end{aligned}$$

Now, it will be found, that where the ratio of increase in the mortality from year to year does not exceed three per cent., or where x does not exceed 1·03, which rarely happens in any table of observations between the ages 15 and 55, the first three terms of the series, formed by multiplying together the factors in the last product, will afford the means of approximating very nearly indeed to the Expectations of life, and the values of Annuities and Assurances on single or on any two joint lives for any term of years, in which the oldest life at the end of the term does not exceed 55 years, or the age to which the above value of x can be used without differing too much from the original observations. After that age, by most tables, the increase in the ratio of mortality from year to year is so rapid, that the same number of terms will not be sufficient to give results with the desired accuracy. The formula for the continuation of the series from age 55 to the oldest life in the Table, or in cases where three lives are involved, must be reserved for another paper.

The product $\left(1-mx^{p}\right)\cdot\left(1-mx^{p+1}\right)\cdot\left(1-mx^{p+2}\right)\cdot\cdot\cdot\cdot$ $\left(1-mx^{p+n-1}\right)$ may be resolved into the following series:

$$\begin{array}{lclcl}
1-mx^{p} & & & & \\
\hline
-mx^{p+1} & + & m^{2}x^{2p+1} & & \\
\hline
-mx^{p+2} & + & m^{2}x^{2p+2} & & \\
 & + & m^{2}x^{2p+3} & - & m^{3}x^{3p+3}\\
\hline
-mx^{p+3} & + & m^{2}x^{2p+3} & - & m^{3}x^{3p+4}\\
 & + & m^{2}x^{2p+4} & - & m^{3}x^{3p+5}\\
 & + & m^{2}x^{2p+5} & - & m^{3}x^{3p+6}\\
\hline
-mx^{p+4} & + & m^{2}x^{2p+4} & - & m^{3}x^{3p+5}\\
 & + & m^{2}x^{2p+5} & - & m^{3}x^{3p+6}\\
 & + & m^{2}x^{2p+6} & - & m^{3}x^{3p+7}\\
 & + & m^{2}x^{2p+7} & - & m^{3}x^{3p+7}\\
 & & & - & m^{3}x^{3p+8}\\
 & & & - & m^{3}x^{3p+9}\\
\hline
\end{array}$$

$$\begin{array}{lll}
-\,mx^{p+n-1}+ & m^2x^{2p+n-1} & -\;m^3x^{3p+n} \\
+ & m^2x^{2p+n} & -\;m^3x^{3p+n+1} \\
+ & m^2x^{2p+n+1} & -\;m^3x^{3p+n+2} \\
 & \text{to} & \quad\text{to} \\
+ & m^2x^{2p+2n-3} & -\;m^3x^{3p+3n-6} \quad \&\text{c.}
\end{array}$$

The general formula for the summation of these series, so that one term may be derived from the other (without the necessity of continued multiplication), to any given number of terms required, will be

$$1-\sum m.x^p.x^n+\sum m^2.x^{2p}.\left(\frac{x^n-1}{x-1}\right)x^n-\sum .m^3.x^{3p}.\left(\frac{x^{2n}-x^2}{(x^2-1)(x-1)}-\frac{x^n-x}{\overline{x-1}^2}\right).x^n \&\text{c.}$$

$$\text{From}\ \underbrace{n=0 \qquad\qquad n=1 \qquad\qquad\qquad\qquad n=2}$$

to $n =$ one less than the number of factors in the given product,

in which it will be perceived that the part within ( ) in each term is the sum of the variable portion under the index n of the preceding term. It will only be necessary to take the first three terms in the summation of the separate series (omitting those in which higher powers than m^2 are involved), which, according to the preceding formula, will evidently be,

$$1-m.x^p.\left(\frac{x^n-1}{x-1}\right)+m^2.x^{2p}.\left(\frac{x^{2n}-x^2}{(x^2-1)(x-1)}-\frac{x^n-x}{\overline{x-1}^2}\right)$$

to obtain a very near approximation to the probability of a life at any given age, living to the end of n years, from that age, within the limits before mentioned.

If m be assumed as the ratio of mortality at any standard age, as for instance the age 15, and x the ratio of increase of mortality from year to year, both these values being determined by trials so as to agree as nearly as possible with the original facts observed, then $\frac{m}{x-1}$, $\frac{m^2x^2}{(x^2-1)(x-1)}$ and $\frac{m^2x}{\overline{x-1}^2}$ are constants, which being once calculated may be used throughout the Table both for single and joint lives, and we will call them

$$\frac{m}{x-1} = \text{X}$$

$$\frac{m^2x^2}{(x^2-1)(x-1)} = \text{Y}$$

$$\frac{m^2x}{\overline{x-1}^2} = x.\text{X}^2 = \text{Z}$$

and the formula for the probability of living n years then becomes,

Formula (1) $1+\text{Y}\,.\,x^{2p}\,(x^{2n-2}-1)-\left(\text{X}.x^p.(x^n-1)\,.+\text{Z}x^{2p}.\,(x^{n-1}-1)\right)$

or $\left(1+\text{X}.x^p+(\text{Z}-\text{Y})\,x^{2p}\right)+\text{Y}\,.\,x^{2p}.\,x^{2n-2}-\left(\text{X}x^p+\text{X}^2x^{2p}\right)x^n$

If $p = 0$, or the given life be that of the standard age of the Table, the formula may be further reduced to

Formula (2) $\left(1+\text{X}+\text{Z}-\text{Y}\right)+\text{Y}.x^{2n-2}-(\text{X}+\text{X}^2)\,x^n$

in which the parts of the expression within ( ) being once computed, will serve for obtaining the probabilities of the same life living to any age thereafter, and may be called V and V′.

It will be convenient to introduce here the short table containing the Constants, the values of which are calculated for three Tables of Mortality, from which examples are hereafter given to show how closely the results approach to the values obtained from Tables corrected by the law of geometrical progression, as well as from the original facts.

Logarithms and Values of Constants in the Formulæ proposed.

Symbols for Probabilities of Living.	EDMONDS'S TABLES.				EQUITABLE EXPERIENCE TABLE "A."	
	Mean Mortality. $x = 1{\cdot}03 = \lambda\ {\cdot}0128372$ Age 12. $m = .00637 = \lambda\ \bar{3}{\cdot}8041394$		Village or Carlisle. $x = 1{\cdot}03 = \lambda\ {\cdot}0128372$ Age 12. $m = {\cdot}005321 = \lambda\ \bar{3}.726$		$x = 1{\cdot}03 = \lambda .0128372$ Age 15. $m = {\cdot}00575 = \lambda . \bar{3}.7596678$	
	Log.	No.	Log.	No.	Log.	No.
$X = \frac{m}{x-1}$	$\bar{1}{\cdot}3270181$	·212333	$\bar{1}{\cdot}2488787$	·177369	$\bar{1}{\cdot}2825465$	·191667
$Y = \frac{m^2 x^2}{(x^2-1)(x-1)}$	$\bar{2}{\cdot}3722146$	·023562	$\bar{2}{\cdot}2159358$	·016441	$\bar{2}{\cdot}2832714$	·019199
$xX^2 = Z = \frac{m^2 x}{\overline{x-1}^2}$	$\bar{2}{\cdot}6668734$	·046438	$\bar{2}{\cdot}5105946$	·032404	$\bar{2}{\cdot}5779302$	·037838
Where $p = o$ $(1+X+Z-Y) = V$	0·0917398	1·235207	0·0767612	1·193332	0·0828951	1·210306
$\left(X + X^2\right) = V'$	$\bar{1}.4106389$	·257418	$\bar{1}{\cdot}3197908$	·208829	$\bar{1}{\cdot}3587000$	·228402
Additional Symbols for Expectation of Life.						
$X' = X . \frac{x}{x-1}$	0·8627340	7·290108	0·7845946	6·089682	0·8182624	6·580553
$Y' = Y . \frac{1}{x^2-1}$	$\bar{1}{\cdot}5875973$	·386899	$\bar{1}{\cdot}4313185$	·269969	$\bar{1}{\cdot}4986541$	·315249
$Z' = Z . \frac{1}{x-1} = X^2 . \frac{x}{x-1}$	0·1897521	1.547933	0·0334733	1·080123	0·1008089	1·261273

Both Mr. Edmonds and Mr. Farr have published sets of Tables computed on the hypothesis of a geometrical progression in the rate of mortality, and to these I shall have recourse to prove the accuracy of the simple formula here suggested. I have slightly altered the table given by the former, so that $x = 1{\cdot}03$ instead of $1{\cdot}0299117$, which will scarcely affect the results in any perceptible degree, if the value of m be adjusted accordingly.

I shall add a few examples founded on the same law, applied to the Experience of the Equitable Society, m and x being deduced from the original observations and the results compared with the corrected Table A, as printed for the use of the members of the Equitable Society by Mr. Arthur Morgan.

In deducing the log. of $\overline{x-1}$ from the log. of x, I have used Mr. Gray's valuable "Tables and Formulæ for the Computation of Life Contingencies," published in 1849, a work which will be found of the utmost importance in calculations of this nature, and it may be also remarked that to use the same Tables $\left(\mathrm{X}x^{p} + \mathrm{X}^{2}x^{2p}\right)$ in Formula (1), and wherever it may hereafter occur may be converted into $\mathrm{X}x^{p}.\left(1 + \mathrm{X}x^{p}\right)$

Example 1.—From Mr. Edmonds' Table of Mean Mortality.

Let it be required to find the probability of a life, age 12, living to age 42.

Here (age 12 being the standard age) $p = 0$ and $n = 30$.

$\mathrm{V} = \left(1 + \mathrm{X} + \mathrm{Z} - \mathrm{Y}\right) = 1{\cdot}2352$

$\mathrm{Y} = \lambda\ \bar{2}{\cdot}37221$

$5{\cdot}5534 = x^{58} = \lambda\ {\cdot}74456$

$\lambda\ 1{\cdot}11677 = {\cdot}1308$

$1{\cdot}3660$

$- {\cdot}6248$

Age 12. Probability of living 30 years $\}$ $\cdot 7412$

$\mathrm{V}' = \left(\mathrm{X}+\mathrm{X}^{2}\right) = \lambda\ \bar{1}{\cdot}41064$

$2{\cdot}4273 = x^{30} = \lambda\ {\cdot}38512$

$\lambda\ \bar{1}{\cdot}79576 = {\cdot}6248$

By Edmonds ·7357
Difference + ·0055

Example 2.—By the same Table, find the probability of a life, age 22, attaining to age 52.

Here Formula (1) must be used. $p = 10$ $n = 30$.

$$1 + \mathrm{Y}\, x^{20} \left(\overset{58}{x-1}\right) - \left(\mathrm{X}.\, x^{10}.\left(\overset{30}{x-1}\right) + \mathrm{Z}.\, x^{20}.\left(\overset{29}{x-1}\right)\right)$$

$\mathrm{Y} = \lambda.\ \bar{2}{\cdot}37221$

$x^{20} = 1{\cdot}8061 = \lambda\ {\cdot}25674$

$\left(\overset{58}{x-1}\right) = 4{\cdot}5534 = \lambda.\ {\cdot}65834$

1.

$\lambda\ \bar{2}.37221 = .1938$

1.1938

.5211

Age 22. Probability of living 30 years $\}$ .6727

$\mathrm{X} = \lambda\ \bar{1}{\cdot}32702$

$x^{10} = 1{\cdot}3439 = \lambda\ {\cdot}12837$

$x^{30} = 1{\cdot}4273 = \lambda\ {\cdot}15452$

$\lambda\ \bar{1}{\cdot}60991 = {\cdot}4073$

$\mathrm{Z} = \lambda\ \bar{2}.66687$

$x^{20} = \lambda\ {\cdot}25674$

$\left(\overset{29}{x-1}\right) = 1{\cdot}3566 = \lambda\ .13245$

$\lambda\ \bar{1}.05606 = .1138$

$- .5211$

By Edmonds' .6623
Difference + .0104

It will be unnecessary to repeat the whole of the process in further examples; but the following results from the other tables, of which the constants are given, will shew the value of the approximation in other cases.

	Age.	Age to be attained.	Probability by the approx.	Difference from	
				Edmonds.	Milne.
EXAMPLE 3. Edmonds' Village or Carlisle Table	12	42	·7777	+ ·0034	+ ·0059
EXAMPLE 4. Ditto.	22	52	·7156	+ ·0062	+ ·0035

In correcting the Table of the Equitable Experience, by assuming $x = 1{\cdot}03$. I have used ·00575 as the ratio of mortality at the standard age 15. In the first ten years, this will give rather too favourable a rate of mortality, when compared with the original facts; but any other mode of correction by regular progression would produce a similar error, and the difference is not important. The total ratio of mortality to a unit of a life at every age, from 15 to 55, will, by this hypothesis, be = ·43255; by the original observations ·42523, show-only a difference of 73 deaths in the whole period in 10,000 living at each age, and agreeing, at every interval after the first ten years, very closely with the observed facts.

	Age.	Age to be attained.	Probability by the approx.	Difference from	
				Orig. facts.	Table A,
EXAMPLE. 5. Equitable Experience	15	55	·6578	+ ·0045	+ ·0001
" 6. Ditto.	35	55	·7579	— ·0028	— ·0108
" 7. Ditto.	15	35	·8568	— ·0020	+ ·0012

In the Registrar General's Sixth Report will be found Mr. Farr's English Life Table, with a number of useful Tables constructed from the corrected data. The mortality, in proportion to the living, obtained from the Census in 1841, appears first to have been corrected by assuming the increase from year to year to be in geometrical progression, and afterwards again correcting the logarithm expressing the number of survivors, at each age, by third differences up to age 55, and by fourth differences after that age.

By the Table for Male lives. } $m = \cdot0069605 = \lambda.\ \bar{3}\cdot8432656$. and $x = \cdot1026011 = \lambda\ \cdot0111522$

Therefore according to our present method of approximation:

$$X = \lambda\ \bar{1}\cdot4281086$$
$$Y = \lambda\ \bar{2}\cdot5718680$$
$$Z = \lambda\ \bar{2}\cdot8673694$$

and by using the same formula as before, the probability of a life, age 15, attaining to 55, will by approximation = ·6218; by Mr. Farr's Table = .6178. The difference therefore is only + .0040.

II. On the probability of Two Joint Lives both living n years.

Let us assume the case of two lives, A and B, whose ages are respectively p and q years older than the standard age, of which the ratio of mortality at any given age may be expressed in the same manner as before, mx^p and mx^q; and the probability of their both living a year at the given age, and at each successive age therefrom, by

$$\begin{array}{llll}
\left(1-mx^{p}\right) & \cdot\left(1-mx^{q}\right) & =1-m\left(x^{p}+x^{q}\right) & +m^{2}.x^{p+q} \\
\left(1-mx^{p+1}\right) & \cdot\left(1-mx^{q+1}\right) & =1-m\left(x^{p+1}+x^{q+1}\right) & +m^{2}.x^{p+q+2} \\
\left(1-mx^{p+2}\right) & \cdot\left(1-mx^{q+2}\right) & =1-m\left(x^{p+2}+x^{q+2}\right) & +m^{2}.x^{p+q+4} \\
\left(1-mx^{p+n-1}\right) & \cdot\left(1-mx^{q+n-1}\right) & =1-m\left(x^{p+n-1}+x^{q+n-1}\right) & +m^{2}.x^{p+q+2n-2}
\end{array}$$

and these successive probabilities being multiplied together for *n* years will give, according to the usual rule, the probability of both the lives attaining to the end of *n* years.

By taking for the approximation proposed only the first three terms, as in the case of single lives, we shall obtain the following series:

$$\begin{array}{ll}
1-m\left(x^{p}+x^{q}\right) & \\
\hline
-m\left(x^{p+1}+x^{q+1}\right) & +m^{2}\left(x^{2p+1}+2\,x^{q+p+1}+x^{2q+1}\right) \\
\hline
-m\left(x^{p+2}+x^{q+2}\right) & +m^{2}\left(x^{2p+2}+2\,x^{q+p+2}+x^{2q+2}\right) \\
 & +m^{2}\left(x^{2p+3}+2\,x^{q+p+3}+x^{2q+3}\right) \\
\hline
 & \text{to} \\
-m\left(x^{p+n-1}+x^{q+n-1}\right) & +m^{2}\left(x^{2p+n-1}+2\,x^{q+p+n-1}+x^{2q+n-1}\right) \\
 & \text{to} \\
 & +m^{2}\left(x^{2p+2n-3}+2x^{q+p+2n-3}+x^{2q+2n-3}\right) \\
\hline
\end{array}$$

The sum of which series

$$=1-m\cdot\left(x^{p}+x^{q}\right)\cdot\left(\frac{x^{n}-1}{x-1}\right)+m^{2}\cdot\overline{x^{p}+x^{q}}|^{2}\cdot\left(\frac{x^{2n}-x^{2}}{\left(x^{2}-1\right)\left(x-1\right)}-\frac{x^{n}-x}{\overline{x-1}|^{2}}\right)$$

From which it evidently appears, that having obtained the constants for the given Table of Mortality, as explained in the article on Single Lives, the approximation to the probabilities of living of two Joint Lives may be calculated with almost equal facility. The formula will be:

$$1+\mathrm{Y}.\overline{x^{p}+x^{q}}|^{2}\cdot\left(x^{2n-2}-1\right)-\left(\mathrm{X}.\left(x^{p}+x^{q}\right)\cdot\left(x^{n}-1\right)+\mathrm{Z}.\,\overline{x^{p}+x^{q}}|^{2}\cdot\left(x^{n-1}-1\right)\right)$$

Example 8.—Edmonds' Mean Mortality. To find the probability that two lives, both aged 12, together attain 30 years.

Here both p and $q=0$ $n=30$

therefore $(x^{p}+x^{q})=2$.

$\mathrm{Y}=\lambda\,\bar{2}.37221$ $\mathrm{X}=\lambda\,\bar{1}{\cdot}32702$

$\left(x^{58}-1\right)=\lambda\ .65834$ $\left(x^{30}-1\right)=\lambda\ {\cdot}15452$

1.

$\lambda\,\bar{1}{\cdot}03055={\cdot}10729\times4={\cdot}4292$ $\lambda\,\bar{1}{\cdot}48154={\cdot}30307\times2={\cdot}60614$

1.4292 $\mathrm{Z}=\lambda.\,\bar{2}{\cdot}66687$

$-$ .8581 $\left(x^{29}-1\right)=\lambda\ {\cdot}13245$

By Edmonds' .5413 .5711 $\lambda\,\bar{2}{\cdot}79932=.063\times4$ .25200

Difference + .0298 $-$ .85814

	Ages.	Probability of attaining to	By approximation.	Difference from Edmonds'
EXAMPLE 9.—Edmonds' Mean Mortality . . .	22-22	52-52	.5054	+ .0668
EXAMPLE 10.—Edmonds' Village or Carlisle . .	12-12	42-42	.6173	+ .0177
				Difference from Original Facts.
EXAMPLE 11.—Equitable Experience.	15-15	45-45	.5972	+ .0046
EXAMPLE 12.—Ditto . .	25-25	55-55	.5254	+ .0271

If these differences should appear too great, it must be remembered that they principally arise in the last few terms of these long series; and in the summation of the series for the Expectations of Life or the values of Life Annuities they will in a great measure disappear.

III.—On the Expectation of Life by the Approximation.

The Curtate Expectation is the sum of the probabilities of living 1, 2, 3, &c. years, to which half a year being added, the true Expectation will be obtained.

If in the formula

$$1 + Y.x^{2p}.\left(x^{2n-2}-1\right) - \left(Xx^{p}.\left(x^{n}-1\right) + Zx^{2p}.\left(x^{n-1}-1\right)\right)$$

n be taken successively equal to 1, 2, 3, &c., the sum of these expressions for n terms, will give the Expectation for n years of a life whose ratio of mortality is mx^{p}.

The formula for the Expectation of Life by the proposed Approximation, therefore becomes,

$$n\left(1+X.x^{p}+\left(Z-Y\right)x^{2p}\right)+Yx^{2p}.\left(\frac{x^{2n}-1}{x^{2}-1}\right)-\left(Xx^{p}+X^{2}x^{2p}\right)\frac{.x.}{x-1}\left(x^{n}-1\right)$$

And if $Y.\frac{1}{x^{2}-1}$ be computed and called Y'

$X.\frac{x}{x-1}$ " " X'

$X^{2}.\frac{x}{x-1}$ " " Z'

These values will also form constants, which may be arranged in a small table, and will serve for all the calculations of the Expectation of Single or two Joint Lives by the approximation. The formula then becomes

Formula (3) $n\left(1+Xx^{p}+\left(Z-Y\right)x^{2p}\right)+Y'.x^{2p}\left(x^{2n}-1\right)-\left(X'x^{p}+Z'x^{2p}\right).\left(x^{n}-1\right)$

And where $p=0$ or at the Standard age of the table

Formula (4) $n\left(V\right)+Y'.\left(x^{2n}-1\right)-\left(X'+Z'\right).\left(x^{n}-1\right)$

Example 13.—Edmonds' Mean Mortality at age 12, find the Expectation of Life for 30 years. $p=o$ $n=30$

$(V) = 1\cdot235207$
$n = 30$
$37\cdot0562$
$+ \ 1\cdot8926$
$38\cdot9488$
$- \ 12\cdot6150$
$26\cdot3338$

$Y' = \lambda.\bar{1}\cdot58760$
$(x^{60}-1) = \lambda.0\cdot68945$
$\lambda \ \cdot27705 = 1\cdot8926 +$
$(X'+Z') = 8.83804 = \lambda.0\cdot94636$
$(x^{30}-1) = 1.4273 = \lambda.0\cdot15452$
$\lambda 1\cdot10088 = 12\cdot6150 -$

By Edmonds' 26·2825
Difference + ·051

Example 14.—By the same Table of Observations, find the Expectation of a Life, age 22, for 30 years. $p=10$ $n=30$

$X = \lambda.\bar{1}\cdot32702$
$x^{10} = 1.3439 = \lambda.\ \cdot12837$
$1\cdot$
$\lambda \ \bar{1}\cdot45539 = \cdot28536$

$(Z-Y) = .022876 = \lambda.\bar{2}\cdot35938$
$x^{20} = 1.8061 = \lambda.\ \cdot25674$
$\lambda \ \bar{2}\cdot61612 = \cdot04132$

$30 \times 1\cdot32668 = 39\cdot8004$
$+ \ 3\cdot4181$
$43\cdot2185$
$-17\cdot9739$
$25\cdot2446$

By Edmonds' 25·1624
Difference + ·082

$Y' = \lambda.\bar{1}\cdot58760$
$(x^{60}-1) = \lambda.\ \cdot68945$
$x^{20} = \lambda.\ .25674$
$\lambda.\ \cdot53379 = 3\cdot4181$

$X' = \lambda.\ \cdot86273$
$x^{10} = \lambda.\ \cdot12837$
$(x^{30}-1) = \lambda.\ \cdot15452$
$\lambda.1\cdot14562 = 13\cdot9836$

$Z' = \lambda.\ \cdot18975$
$x^{20} = \lambda.\ \cdot25674$
$(x^{30}-1) = \lambda.\ \cdot15452$
$\lambda \ \cdot60101 = 3\cdot9903$

$$\left.\begin{array}{r} 13\cdot9836 \\ 3\cdot9903 \end{array}\right\} \quad 17\cdot9739$$

	Age			By Approx.	Diff.
Example 15.—By Edmonds' Village or Carlisle. Expectation for 30 years	12	=	26.853	26.841	— .012
Example 16.—Ditto	22	=	25.885	25.835	— .050

The Expectations of two Joint Lives whose ratios of mortality, at their given ages, are, respectively mx^p and mx^q, may be found by substituting in the preceding formula (x^p+x^q) wherever x^p occurs.

IV.—On the Values of Annuities upon Single and Two Joint Lives by the Approximation.

Let v^n = the present value of £1 due n years hence; then, by turning to Article I., the value of £1 receivable provided a life at any given age, p years older than the Standard Age, be existing n years hence, will be found

$$=v^n\left(1+\mathrm{X}x^p+(\mathrm{Z}-\mathrm{Y})x^{2p}+\mathrm{Y}x^{2p}x^{2n-2}-(\mathrm{X}.x^p+\mathrm{X}^2x^{2p})x^n\right)$$

and the sum of the expressions, from $n=1$ up to $n=$ the given term of years, will be the value of an Annuity of £1 per annum for n years on the given life.

But $\sum.v^n$ from $n=1$ to $n=$ the given term, is the value of an Annuity certain for n years, which, if we denote by I_n, the formula for n years will be,

Formula (5.)

$$\mathrm{I}_n\left(1+\mathrm{X}x^p+(\mathrm{Z}-\mathrm{Y})x^{2p}\right)+\mathrm{Y}.x^{2p}v.\left(\frac{\overline{x^2v}|^n-1}{x^2v-1}\right)-(\mathrm{X}x^p+\mathrm{X}^2x^{2p})xv.\left(\frac{\overline{xv}|^n-1}{xv-1}\right)$$

Formula (6.)

Where $p=o$, or at the Standard Age the formula may be reduced to

$$\mathrm{I}_n.\mathrm{V}+\mathrm{Y}.v.\left(\frac{\overline{x^2v}|^n-1}{x^2v-1}\right)-\mathrm{V}'.xv\left(\frac{\overline{xv}|^n-1}{xv-1}\right)$$

Before proceeding to give examples, I may observe, that the advantage of finding x to represent 1.03, (which it does in the tables we have been using,) will be seen, by its allowing of great facility in the calculations of the Values of Annuities on Single and Joint Lives at 3 per cent., which is a very common rate of interest in office calculations. For in this case it is evident that as $v.=\frac{1}{1.03}$ and $x=1.03$, $xv=1$, and, therefore, $xv.\left(\frac{\overline{xv}|^n-1}{xv-1}\right)=n$; being the sum of n terms of a series, the first term of which is 1, and the common ratio 1. The Formula, then, at the Standard age, becomes for Single Life Annuities at 3 per cent.

$$\mathrm{I}_n.\mathrm{V}+\mathrm{Y}.v.\left(\frac{x^n-1}{x-1}\right)-\mathrm{V}'.n$$

EXAMPLE 17.—Edmonds' Mean Mortality. Find the value of an Annuity, at 3 per cent, for 30 years, on a life, age 12.

Here $p=o$, $n=30$, and as $x=v$, the last Formula may be used.

$\mathrm{V}=1{\cdot}23521=\lambda.\ {\cdot}09174$

$\mathrm{I}_n=19{\cdot}6004=\lambda.1.29226$

$1.38400=24{\cdot}2105$

$1{\cdot}0344$

$25{\cdot}2449$

$7{\cdot}7226$

$17{\cdot}5223$

$\mathrm{Y}=\lambda\bar{2}{\cdot}37221$

$v=\lambda\bar{1}{\cdot}98716$

$\left(\frac{x^{30}-1}{x-1}\right)=\lambda 1{\cdot}65532=45{\cdot}2188$

$\lambda\ {\cdot}01469=1{\cdot}0344$

$\mathrm{V}'.n.={\cdot}25742\times30=7{\cdot}7226$

By Edmonds'	17·549
Difference	—·027

$\left(\frac{x^{30}-1}{x-1}\right)$=Amount of £1 per annum, in 29 years at 3 per cent, Compound Interest.

Example 18.—Edmonds' Mean Mortality. Find the value of an Annuity, at 4 per cent., for 30 years, on a life age 22.

$$xv = \lambda.1{\cdot}995804$$

$$x^2v = \lambda.\ {\cdot}008641$$

In this case Formula (5) must be used, $p=10$ $v=\frac{1}{1.04}$ $n=30$

By Example 14.

$\left(1+Xx^{10}+(Z-Y)x^{20}\right)=1{\cdot}32668=\lambda.\ {\cdot}12277$

$I_{30}=17{\cdot}292=\lambda.1{\cdot}23785$

$22{\cdot}9414=\lambda.1{\cdot}36062$

$+\ 1{\cdot}6624$

$24{\cdot}6038$

$-\ 9{\cdot}5065$

$15{\cdot}0973$

By Edmonds' 15·063

Difference + ·034

$Y=\lambda.\bar{2}{\cdot}37221$

$x^{20}=\lambda.\ {\cdot}25674$

$\frac{v}{x^2v-1}=\frac{1}{{\cdot}0209}=\lambda.1{\cdot}67985$

$\overline{x^2v}|^{30}-1=\lambda.\bar{1}{\cdot}97195$

$1{\cdot}6624=\lambda.\ {\cdot}22075$

$X.x^{10}=\lambda\,\bar{1}{\cdot}45539$

By Gray $\left(1+Xx^{10}\right)=\lambda\ {\cdot}10902$

$\lambda\,\bar{1}{\cdot}56441$

$xv=\lambda\,\bar{1}{\cdot}99580$

$\frac{1}{(xv-1)}=\lambda\,2{\cdot}01704$

$\overline{xv}|^{30}=\bar{1}.87412$

By Gray $\left(\overline{xv}|^{30}-1\right)=\bar{1}{\cdot}40076$

$9.5065=\lambda{\cdot}97801$

For the values of Annuities on two joint lives it will be unnecessary to repeat the reasoning; for by discounting the probabilities of both the lives attaining 1, 2, 3, &c., years, in the same manner as we have just done for the single lives, it will be seen almost by inspection, that it requires only to substitute $\left(x^p+x^q\right)$, wherever x^p occurs in the formula for the Single lives, to obtain the value of the Annuity for n years on the Joint lives.

Example 19.—Edmonds' Mean Mortality. Find the value of an Annuity at 3 per cent. for 30 years, on two joint lives, both age 12.

Here $p=o$ and $q=o$, therefore $\left(x^p+x^q\right)=2$

$v=1.03$ $n=30$

1.

$2X={\cdot}42466$

$4(Z\text{-}Y)\ {\cdot}09150$

$1{\cdot}51616=\lambda\ {\cdot}18074$

$I_{30}=19{\cdot}6004=\lambda\,1{\cdot}29226$

$29{\cdot}7167=\lambda.\bar{1}{\cdot}47300$

$4{\cdot}1376$

$33{\cdot}8543$

$18{\cdot}1500$

$15{\cdot}7043$

By Edmonds 15·8205

Difference — ·106

$4Y={\cdot}094248=\lambda\bar{2}{\cdot}97427$

$v=\frac{1}{1.03}=\lambda\bar{1}{\cdot}98716$

$\frac{x^{30}-1}{x-1}=45{\cdot}2188=\lambda 1{\cdot}65532$

$4{\cdot}1376=\lambda\ {\cdot}61675$

$2X\ {\cdot}42466$

$4X^2={\cdot}18034$

${\cdot}60500$

$xv\left(\frac{\overline{xv}|^{30}-1}{xv-1}\right)=n\ \ 30$

$18{\cdot}1500$

V. On Annuities on the longest of two lives for a term of n years by the approximation.

Having already shown the method of approximating to the value of an Annuity on the two single lives, and also to the value of an Annuity on the joint duration of two lives, we may thence deduce the formula for the value of an Annuity for the longest of the two lives during the given term by taking the latter value from the sum of the two former, thus, (representing for greater simplicity $v.\left(\frac{\overline{x^2 v}|^{n}-1}{x^2 v-1}\right)$ by=T, and $xv\left(\frac{\overline{xv}|^{n}-1}{xv\;-1}\right)$ by T′). A the life whose ratio of mortality is mx^p and B's$=mx^q$

$$\left.\begin{array}{r}\text{Art. 4. Value of Ann.}\\ \text{on A's life}\end{array}\right\}=\mathrm{I}_n\left(1+\mathrm{X}x^p+(\mathrm{Z}-\mathrm{Y})x^{2p}\right)+\mathrm{Y}x^{2p}\mathrm{T}-\left(\mathrm{X}x^p+\mathrm{X}^2x^{2p}\right)\mathrm{T}'$$

$$\left\{\begin{array}{c}\text{By Formula (5)}\\ \text{Value of Annuity on}\\ \text{B's life..........}\end{array}\right\}\underline{=\mathrm{I}_n\left(1+\mathrm{X}x^q+(\mathrm{Z}-\mathrm{Y})x^{2q}\right)+\mathrm{Y}x^{2q}\mathrm{T}-\left(\mathrm{X}x^q+\mathrm{X}^2x^{2q}\right)\mathrm{T}'}$$

Sum of both values—

$$=\mathrm{I}_n\left(2+\mathrm{X}.(x^p+x^q)+(\mathrm{Z}-\mathrm{Y})(x^{2p}+x^{2q})\right)+\mathrm{Y}.(x^{2p}+x^{2q})\mathrm{T}\Big)-\left(\begin{array}{c}\mathrm{X}\left(x^p+x^q\right)\\ +\mathrm{X}^2\left(x^{2p}+x^{2q}\right)\end{array}\right)\mathrm{T}'$$

From which, deducting the value by approximation of the Annuity on the joint lives—

$$=\mathrm{I}_n\left(1+\mathrm{X}.(x^p+x^q)+(\mathrm{Z}-\mathrm{Y}).\overline{x^p+x^q}|^2\right)+\mathrm{Y}.\overline{x^p+x^q}|^2.\mathrm{T}-\left(\begin{array}{c}\mathrm{X}\left(x^p+x^q\right)\\ +\mathrm{X}^2.\left(x^p+x^q\right)^2\end{array}\right)\mathrm{T}'$$

Formula (7)

We obtain for the approximation to the value of an Annuity on the Longest of two lives—

$$=\mathrm{I}_n\left(1-(\mathrm{Z}-\mathrm{Y}).2.x^{p+q}\right)-\mathrm{Y}.2.x^{p+q}\mathrm{T}+\mathrm{X}^2.2.x^{p+q}\mathrm{T}'$$

Example (20).—Edmonds' Mean Mortality. Find the value of an Annuity on the longest of two lives, both aged 12, for 40 years, at 3 per cent.

Here $p=o$ $q=o$ $v=\frac{1}{103}$ $n=40$, and since $x=1.03$, we may shorten the operation as before noticed.

$$\begin{array}{lrl}
 & 1. & \\
2(\mathrm{Z}-\mathrm{Y})=\cdot022876\times2= & \cdot04575 & \\
 & \overline{} & \\
 & \cdot95425 & =\lambda\bar{1}\cdot97966\\
\mathrm{I}_{10}= & 23\cdot1148 & =\lambda1\cdot36389\\
 & 22\cdot0572 & =\lambda1\cdot34355\\
 & 3\cdot6068 & \\
 & \overline{25\cdot6640} & \\
 & 3\cdot3048 & \\
\text{Answer} & \overline{22\cdot3592} &
\end{array}
\qquad
\begin{array}{l}
2\mathrm{Y}=\cdot047124=\lambda\bar{2}\cdot67324\\
v=\frac{1}{1.03}\quad=\lambda\bar{1}\cdot98716\\
\frac{x^{40}-1}{x-1}=72\cdot2342=\lambda1\cdot85874\\
\lambda\;\cdot51914=-3\cdot3048\\
2\mathrm{X}.^2=\cdot09017\\
xv\left(\frac{\overline{xv}|^{40}-1}{xv-1}\right)=\frac{40}{+3\cdot6068}
\end{array}$$

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VI. Values of Reversions for terms of years by the approximation.

The probability of any life A, whose ratio of mortality is denoted by mx^{p}, failing in any given year n will evidently be equal to that of its attaining one year less than n multiplied by the ratio of mortality at that age, or by mx^{p+n-1}. The general expression for this probability will therefore be (by referring to article I) for the expression of the probability of living to the age $p + n - 1$

$$m.x^{p+n-1}\Big[\Big(1+\mathrm{Y}.\,x^{2p}\,(x^{2n-4}-1)-\Big(\mathrm{X}x^{p}.\;(x^{n-1}-1)+\mathrm{Z}\,x^{2p}(x^{n-2}-1)\Big)\Big]$$

By interpreting n successively by 1, 2, 3, &c. up to the given term, and summing the series, we shall have the sum of the probabilities of dying in each year of the term, or the present value of £1 payable at the end of any year in which the life shall fail during the term, without reckoning interest of money.

By multiplying the above expression by v^{n} we shall obtain the present value of £1 payable in the event of the life failing in any given year n, discounted at the rate of interest represented in v

$$m.x^{p}.x^{n-1}.v^{n}\Big[\Big(1+\mathrm{Y}x^{2p}\,(x^{2n-4}-1)-\Big(\mathrm{X}x^{p}.(x^{n-1})+\mathrm{Z}x^{2p}.(x^{n-2}-1)\Big)\Big]$$

and consequently interpreting n as before, by 1, 2, 3, &c. up to the given term we shall have

$$\sum m.x^{p}.x^{n-1}.v^{n}\Big[\Big(1+\mathrm{Y}x^{2p}.(x^{2n-4}-1)-\Big(\mathrm{X}x^{p}.(x^{n-1}-1)+\mathrm{Z}x^{2p}.(x^{n-2}-1)\Big)\Big)\Big]$$

from $n = 1$ to $n =$ the given term, for the general expression for the value of a reversion on the life of A for the term

$$=mx^{p}v.\left(\frac{\overline{xv}|^{n}-1}{xv-1}\right)\Big(1+\mathrm{X}^{p}+(\mathrm{Z}-\mathrm{Y})x^{2p}\Big)+\mathrm{Y}.x^{3p}.\frac{mv}{x^{2}}\left(\frac{\overline{x^{3}v}|^{n}-1}{x^{3}v-1}\right)-\Big(\mathrm{X}x^{2p}+\mathrm{X}^{2}x^{3p}\Big).mv.\left(\frac{\overline{x^{2}v}|^{n}-1}{x^{2}v-1}\right)$$

Formula (8)

$$=mx^{p}.v\left[\left(\frac{\overline{xv}|^{n}-1}{xv-1}\right).\Big(1+\mathrm{X}x^{p}-(\mathrm{Z}-\mathrm{Y})x^{2p}\Big)+\mathrm{Y}\frac{x^{2p}}{x^{2}}\left(\frac{\overline{x^{3}v}|^{n}-1}{x^{3}v-1}\right)-\Big(\mathrm{X}x^{p}+\mathrm{X}^{2}x^{2p}\Big).\left(\frac{\overline{x^{2}v}|^{n}-1}{x^{2}v-1}\right)\right]$$

Where $p = o$.

Formula (9) $$m.v.\left[\frac{\overline{xv}|^{n}-1}{xv-1}.\,\mathrm{V}+\mathrm{Y}\frac{1}{x^{2}}.\left(\frac{\overline{x^{3}v}|^{n}-1}{x^{3}v-1}\right)-\mathrm{V}'.\left(\frac{\overline{x^{2}v}|^{n}-1}{x^{2}v-1}\right)\right]$$

Example (21).—Edmonds' Table Mean Mortality. Find the value at 3 per cent. of a Reversion of £1 for a term of 40 years on a life age 12.

In this case $p = o$ therefore $1 + \mathrm{X}x^{p} + (\mathrm{Y} - \mathrm{Z})x^{2p} = \mathrm{V}$

and $(\mathrm{X}x^{p}+\mathrm{X}^{2}x^{2p})=\mathrm{V}'$ in the Table of Constants,

$$v = \tfrac{1}{1.03} \qquad n = 40 \qquad xv = 1 \qquad \frac{\overline{xv}|^{40}-1}{xv-1} = 39$$

$$\begin{array}{rl} nV = 39 \times .1\cdot23521. = & 48\cdot1731 \\ + & 3\cdot5158 \\ \hline & 51\cdot6889 \\ - & 18\cdot5943 \\ \hline \lambda 1\cdot51975 = & 33\cdot0946 \\ vm = \lambda \bar{3}\cdot79130 & \\ \hline \text{Answer } \cdot20467 = \bar{\lambda}1\cdot31105 & \end{array}$$

$$\begin{array}{rl} Y & = \lambda\, \bar{2}\cdot37221 \\ \frac{1}{x^{2}} & = \lambda\, \bar{1}\cdot97432 \\ \overline{x}^{2}\big|^{40}_{-1} & = \lambda\, \cdot98412 \\ \left(\frac{1}{x^{2}-1}\right) & = \lambda\, 1\ 21538 \\ \hline & \lambda\, \cdot54603 = 3\cdot5158 \\ \left(X + X^{2}\right) = V' & = \lambda\, \bar{1}\cdot41064 \\ \frac{x^{40}-1}{x-1} & = \lambda\, 1\cdot85874 \\ \hline & \lambda\, 1\cdot26938 = 18\cdot5943 \end{array}$$

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Examples might be given from other tables, and the formula for Survivorship Annuities and Assurances deduced by the same reasoning. But these, together with the more difficult application of the proposed method to ages at which the increase of mortality is so much more rapid, namely, at 55 and upwards, must be reserved for another paper.

S. B.

25th May, 1850.

THE HISTORY OF ASSURANCE.

On indirect Methods of acquiring Knowledge.—The Method of History.—The First Table of Mortality. By Edwin James Farren, Esq.

The term scholar, as current in the English language, has two extreme acceptations, tyro and proficient; or what the later Greeks fancifully termed the alpha and omega of acquirement. If we attempt to trace the steps by which even the adult student of any especial branch of professional or literary knowledge has fairly passed the boundary defined by the one meaning in passing on to that position denoted by the other, it will commonly be found, that in place of that lucid order, that straight line from point to point, which theory and resolve generally premise, the real order of acquirement has been desultory—the real line of progression, circuitous and uncertain. Having, however, by dint of perseverance, achieved so much of the journey as will enable him to look back with something like exultation upon the distance traversed, the traveller can scarcely help imagining that he is now already in a position to indicate to others a road more direct than that by which he has actually passed. He would close up certain bye-ways, would avoid certain windings, and concludes that by such means his own progress might have been materially advanced. As he further progresses, however, and still finds he can proceed but slowly in the direct line, the conviction at last comes, for, in defiance of his own resolves, it is rarely that he can exclude his own example from the category, that certain *indirect* roads and bye-paths, commonly disregarded as divergent and merely tending to unnecessary delay, are really necessary auxiliaries for the proper